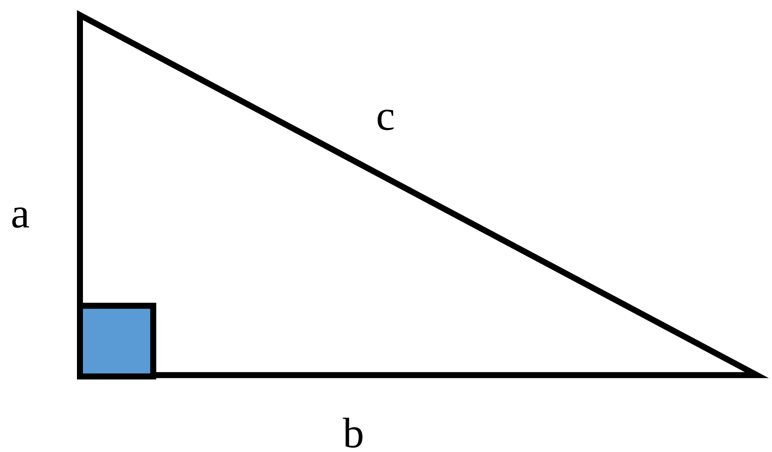
Name ___________________________________ 2020 Version

Algebra 2H Unit "8": Trigonometric Functions and The Unit Circle --- Notes Day 1

So, what is Trigonometry? At its core, Trig (that's a simpler abbreviation that I will use quite often) is the study of the relationships between sides and angles of a triangle. I know that it sounds like Geometry, and there are a lot of tie-ins to Geometry, but it is its own branch of mathematics. We could spend almost an entire year on just investigating Trig and its relationships with both Algebra and Geometry.

There are two main types of Trig: 1) Right Triangle Trigonometry and 2) Non-Right Triangle Trigonometry. We will only be focusing on the first type, involving right triangles. You may be exposed to some non-right triangle stuff once you reach pre-calculus and beyond. But what we do in this unit will serve as a basis for all of that.

In Algebra 1, I am sure you spent a little time with right triangles, specifically the Pythagorean Theorem (which most of you memorized as: $a^2 + b^2 = c^2$). Remember that the "a" and "b" values refer to the legs of the right triangle, the two sides that come together to form the right angle. The "c"-value is the hypotenuse, which is always directly across from the right angle. The "a" and "b" values are interchangeable in the equation, but the "c"-value must always represent the hypotenuse. The picture below should refresh your memory.



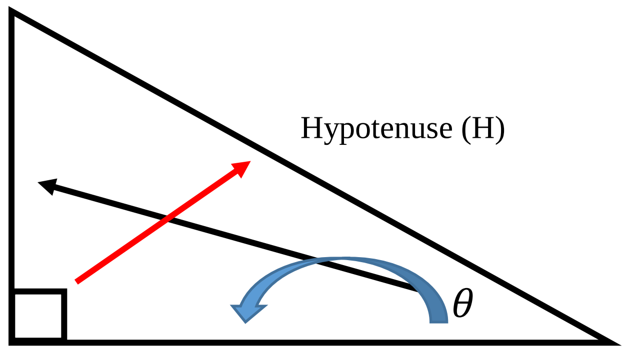
To help illustrate the relationships between the sides and angles, we need to be introduced to three new functions: Sine, Cosine, and Tangent. These are often abbreviated as: Sin, Cos, Tan and these buttons can be found on scientific calculators (even your iPhones if you turn them sideways!) and on graphing calculators like the Inspire.

Each of the three trig functions (Sin, Cos, Tan) represent different side relationships based on a given angle. In trig, you may notice the use of Greek letters to represent angles. Where we might use "x" or "n", trig uses $\theta\ (theta)\ or\ \alpha\ (alpha)$. It's not necessary to know the Greek alphabet, just be ready to see these new symbols as we move forward. I tend to use θ a lot when representing angles…just a habit I developed over many (30+) years of doing this. Angles may also be referred to by their letter.

Are you ready to take on a new challenge…a bit nervous…interested (I hope)? Then here we go…

<u>Part One</u> --- Defining the three Trigonometric Functions (Sin, Cos, Tan)

To understand Trigonometry, we really have to know the three main functions. Each one represents the ratio of sides of a triangle, and in a specific order. The triangle below will help us to understand how each trig function relates to a pair of sides. We will use θ to represent an unknown angle of the triangle.



The Hypotenuse (which we will simply call H) is always the longest side and right across from the right angle. The Opposite side (O), will vary depending on where θ is. If you draw a straight line out from θ, the side you intersect is considered opposite (O) from it. The Adjacent side (A) is the remaining side, and it is always between the angle (θ) and the right angle.

I know, I know…but what does this have to do with Sin, Cos, and Tan? It has to do with an old horse…

Some Old Horse Caught Another Horse Taking Oats Away

Now I know that some of you (most of you?) are rolling your eyes and making fun of me…go ahead, laugh it up! This corny little phrase will be stuck in your head for the rest of your life. Ask your parents if they know SOHCAHTOA or some phrase about a horse…I'll wait…

Bet many of your parents sighed or rolled THEIR eyes, but they still remembered it! This phrase is how we remember which trig function goes with which pair of sides.

<u>S</u>ome <u>O</u>ld <u>H</u>orse <u>C</u>aught <u>A</u>nother <u>H</u>orse <u>T</u>aking <u>O</u>ats <u>A</u>way

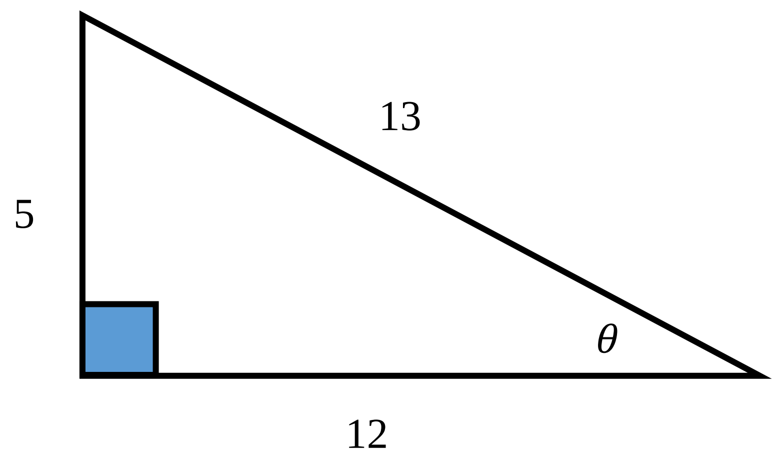
=

<u>S</u>ine <u>O</u>pposite <u>H</u>ypotenuse <u>C</u>osine <u>A</u>djacent <u>H</u>ypotenuse <u>T</u>angent <u>O</u>pposite <u>A</u>djacent

The first letters of each word form a pneumonic that helps us remember the side relationship for each function.

$$\boxed{\begin{aligned} \sin\theta &= \frac{\text{opposite side length}}{\text{hypotenuse length}} \\ \cos\theta &= \frac{\text{adjacent side length}}{\text{hypotenuse length}} \\ \tan\theta &= \frac{\text{opposite side length}}{\text{adjacent side length}} \end{aligned}}$$

Is the confusion setting in yet? I figured…let's try an example to illustrate the functions and see if that helps. The right triangle below has all three side lengths given.

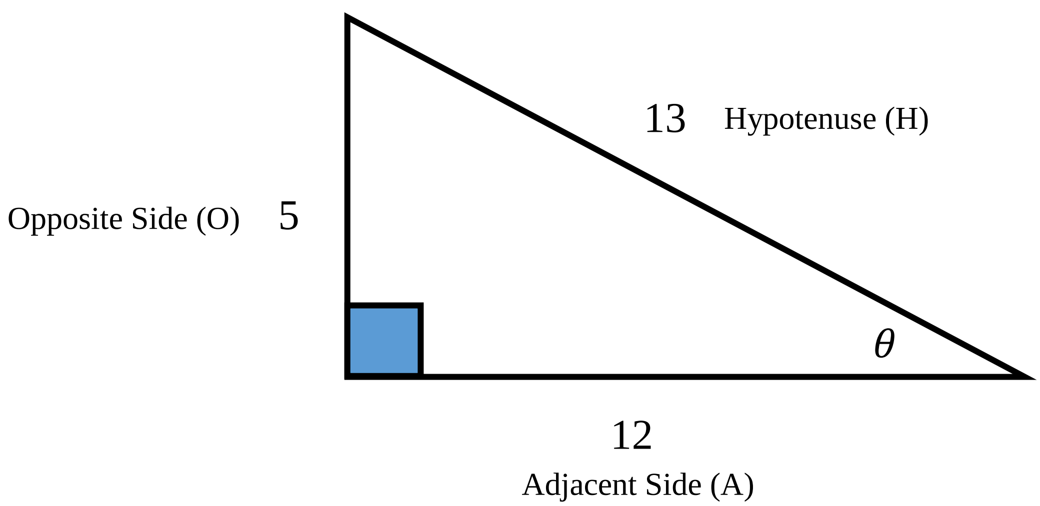


Based on the lengths as given, and SOHCAHTOA (that's the trig relationships), here's how we should attack this:

1) label the three sides with O (Opposite), A (Adjacent) and H (Hypotenuse) with respect to angle θ

2) Find the value (ratio) of the three trig functions

So, adjusting the diagram above, it would now look like this:



So, here are our three trig ratios for this problem: $\sin\theta = \frac{opp}{hyp} = \frac{5}{13}$ $\cos\theta = \frac{adj}{hyp} = \frac{12}{13}$ $\tan\theta = \frac{opp}{adj} = \frac{5}{12}$

Each trig function is represented as the ratio of two sides, depending on where θ lies. Please realize that θ doesn't always have to be in the lower angle. Which means that the O and A can change depending. The H will always remain across from the right angle.

Trig 2020 – Notes Day #2

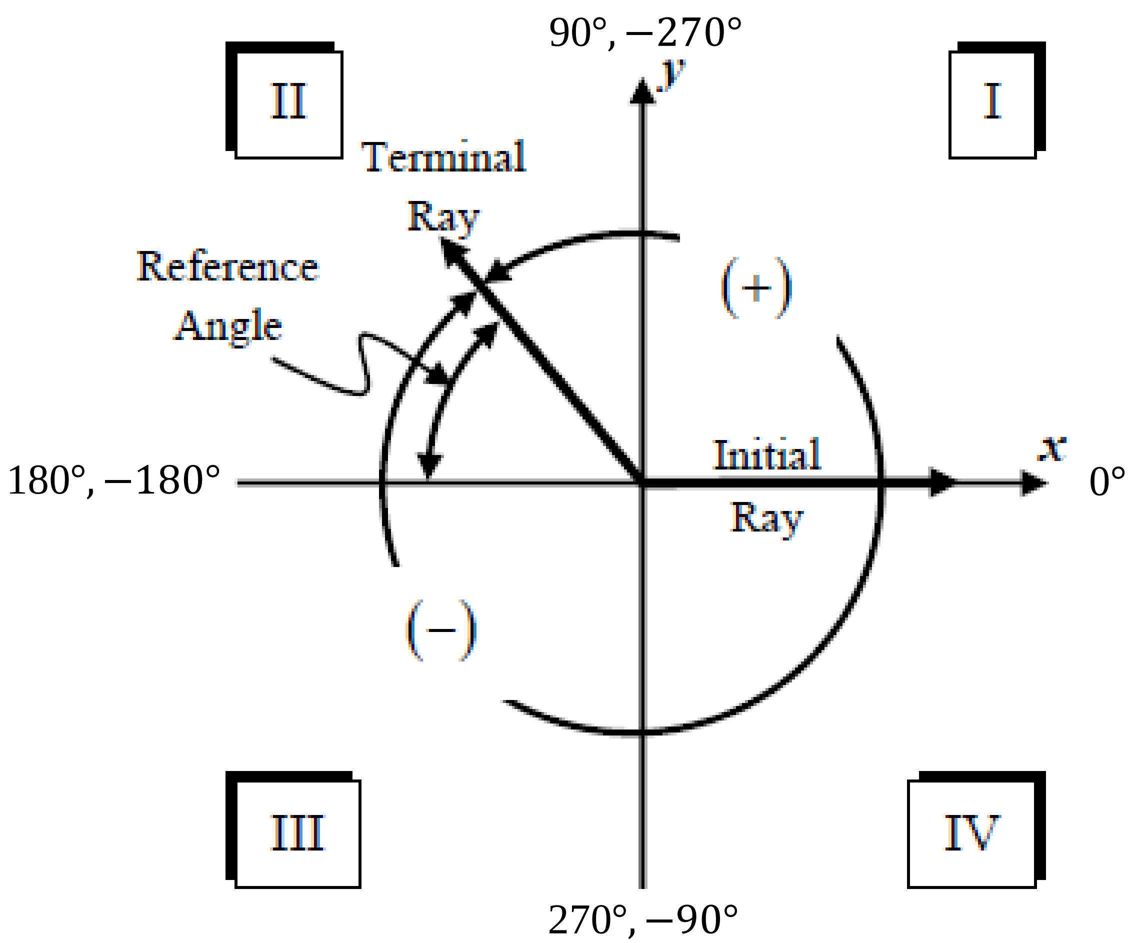
Okay, so we introduced the main three trig functions that we will be working with this unit. These functions are based on the geometry of a circle and rotations around its center. Sometimes the trigonometric functions are known as **circular functions.** In this next lesson, we introduce some basic terminology and concepts concerning angles.

Standard Position: An angle is said to be drawn in standard position if its vertex is at the origin and its initial ray points along the positive *x*-axis.

Positive and Negative Rotations: A rotation is said to be ***positive*** if the initial ray is rotated counterclockwise to the terminal ray and said to be ***negative*** if the initial ray is rotated clockwise to the terminal ray. It may seem counterintuitive that clockwise is negative, but that's the way it is.

Coterminal Angles: Any two angles drawn in standard position that share a terminal ray.

Reference Angles: The positive acute angle formed by the terminal ray and the ***x*-axis**.



Now let's try a few practice questions:

Hint: By either adding (counterclockwise) or subtracting (clockwise) $360°$ we travel a full rotation, thus ending on the same terminal ray (coterminal angles).

Example 1 For each of the following angles, given by the Greek letter **theta**, θ, identify the quadrant that the terminal ray falls in. You can use roman numerals or standard Arabic.

(a) $\theta = 145°$ (b) $\theta = 320°$ (c) $\theta = -210°$

145 is between 90 and 180, so it lies in Quad. 2

320 is between 270 and 360, so it lies in Quad. 4

-210 is between -180 and -270, so it lies in Quad. 2

(d) $\theta = 72°$ (e) $\theta = 250°$ (f) $\theta = -460°$

72 is between 0 and 90, so it lies in Quad. I

250 is between 180 and 270, so it lies in Quad. III

-460 = -100, and is between -90 and -180, so it lies in Quad. III

The negative angles are always harder! Just remember that each group of 360 is one full rotation.

Example 2 In which quadrant would the terminal ray of an angle drawn in standard position fall if the angle measures $860°$?

(1) I (3) III
(2) II (4) IV

If you subtract two groups of 360 (so, 720), you'd have 140…which would fall in Quad. 2

Example 3 Give a negative angle that is coterminal (lands in the exact same spot) with each of the following positive angles, given by the Greek letter **alpha**, α.

(a) $\alpha = 90°$

-270

(c) $\alpha = 330°$

-30

(b) $\alpha = 120°$

-240

(d) $\alpha = 210°$

-150

Do you notice a pattern with the two numbers?

Example 4 Coterminal angles drawn in standard position will always have measures that differ by an integer multiple of

(1) $90°$ (3) $180°$
(2) $360°$ (4) $720°$

Example 5 For each of the following angles given by the Greek letter **beta**, β, state **beta's** reference angle, β. Remember that a reference angle is the distance from that angle to the x-axis. Reference angles are always positive and acute (less than 90).

(a) $\beta = 160°$

180 – 160 = 20

(b) $\beta = 300°$

360 – 300 = 60

(c) $\beta = 210°$

210 – 180 = 30

(d) $\beta = 78°$

Since it's acute, it is already a ref angle.

(e) $\beta = -110°$

180 + (-110) = 70

(f) $\beta = -280°$

360 + (-280) = 80

Now, try the questions on Trig Assignment #2.

Trig 2020 – Notes Day #3

The Unit Circle

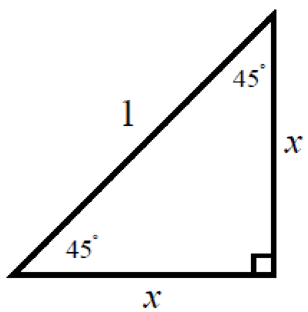
The basis of trigonometry is a very special circle known as ***the unit circle****. This is simply a circle that has its center located at the origin and has a radius equal to one unit (hence the name "unit circle").*

Example 6 From our work with equations of circles, which of the following would represent the equation of the unit circle? Remember: $(x-h)^2+(y-k)^2=r^2$

(1) $x+y=1$ (3) $y=x^2+1$
(2) $x^2+y^2=1$ (4) $(x-1)^2+(y-1)^2=1$

Next, we will seek to produce some of the coordinate points that lie on the unit circle using the Pythagorean Theorem. The next two exercises will illustrate the two important right triangles we will need. You will learn more about these in Geometry, but their introduction now is important to us moving on.

Example 7 Consider the right triangle shown whose hypotenuse is equal to one and whose angles are both equal to 45°. Since this is an isosceles right triangle, the two equal sides are labeled *x*. Solve for *x* and place your answer in simplest radical form.



$x^2+x^2=1^2$ Plug sides into Pythagorean theorem

$2x^2=1$ Combine like terms.

$x^2=\frac{1}{2}$ Divide to solve for x.

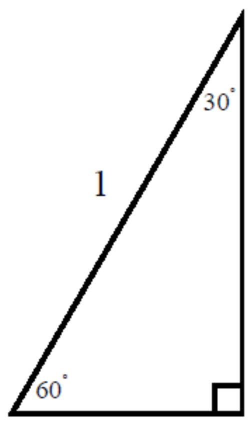
$x=\frac{\sqrt{2}}{2}$ Square root, then rationalize.

Example 8 Consider the $30°-60°$ right triangle shown whose hypotenuse is equal to one. Clearly this triangle is half of an equilateral triangle.

(a) What is the length of the shorter side of this right triangle?

Since all sides of the original equilateral would have to be 1, the short side would become ½.

(b) Using the Pythagorean Theorem, find the length of the longer side in simplest radical form. We'll make the longer leg = x.



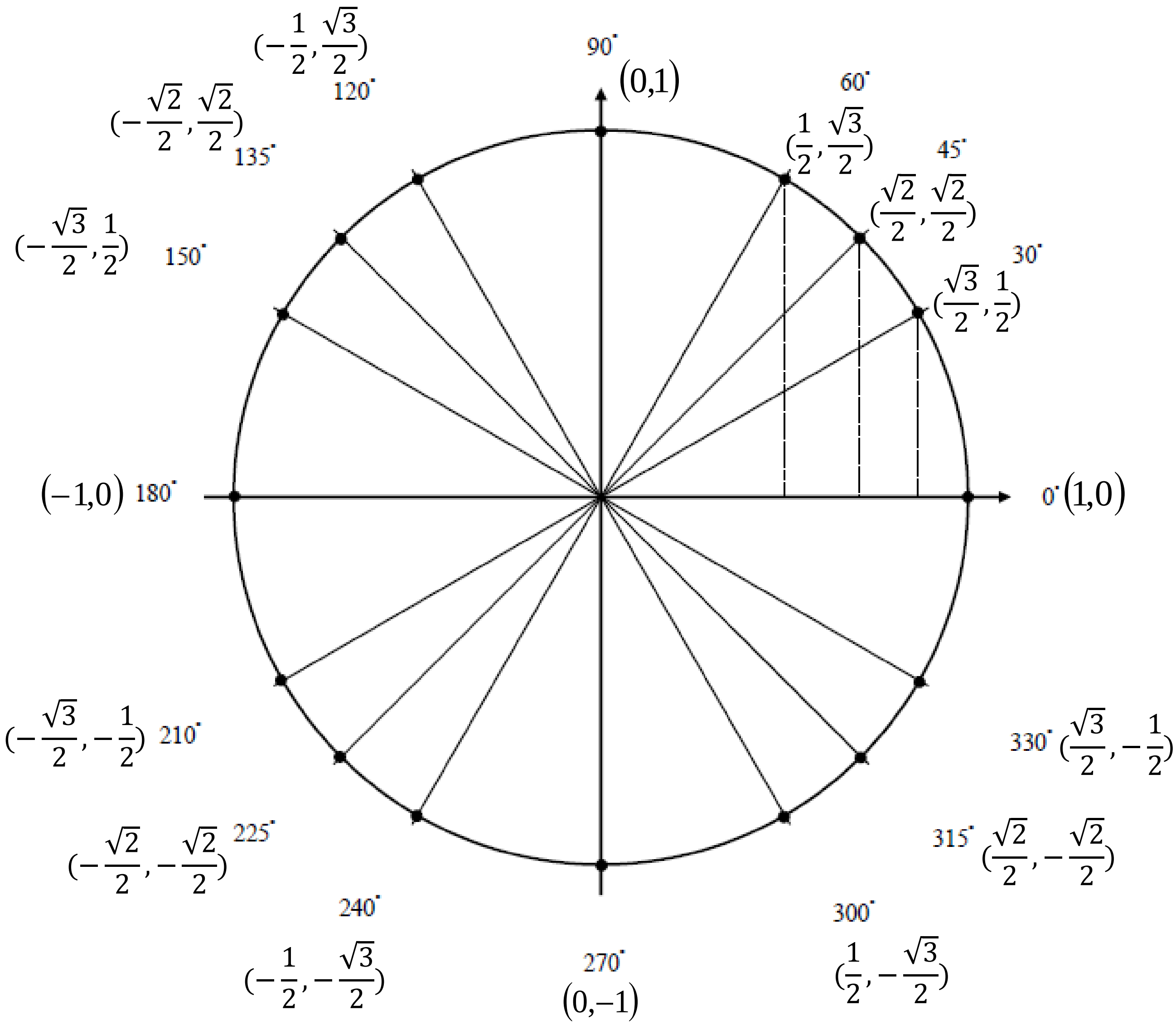
$(\frac{1}{2})^2+x^2=1^2$ Place into Pythagorean Formula.

$\frac{1}{4}+x^2=1$ Simplify.

$x^2=\frac{3}{4}$ Solve for x.

$x=\frac{\sqrt{3}}{2}$ Square root and rationalize.

Example 9 The diagram below represents the **unit circle**. Based on your work from Examples 7 and 8, fill in the ordered pairs at each of the following angles that are assumed to be drawn in standard position. *HINT: Reflect each triangle in quadrant I over the x-axis, y-axis, or both.* Remember that the radius is always 1 unit, so when we look at the angles, we'll be using the values we just solved for in the last two examples.



So, for the triangle that is at 30°, we would use the 30-60-90 right triangle. According to example 8, this means that the shorter side is ½ (which is the y-coordinate) and the longer side (along the x-axis) is $\frac{\sqrt{3}}{2}$.

The triangle that is at 45° is a little easier, since it's an isosceles right triangle (45-45-90). Both of the legs of the triangle would be the same, and according to example 7, they would both be $\frac{\sqrt{2}}{2}$.

The third triangle (the one at 60°), would just be the reverse of the first one, since the two legs are flipped. Hence, the two coordinates have switched places.

Your assignment is to fill in the rest of the labeled angles, by reflecting the three triangles over the axes.

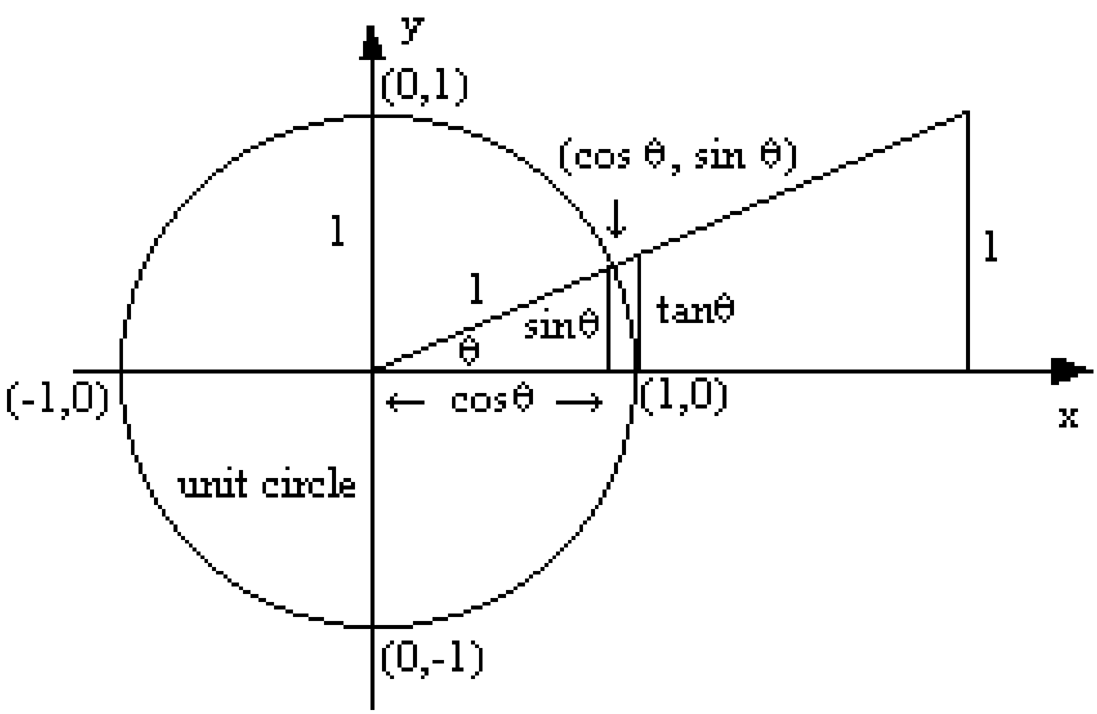
<u>Trig 2020 – Notes Day #4</u>

Hopefully you took notice of the fact that although the values for the x and y-coordinates were similar in each quadrant, they should have been affected a little. I am referring to whether they were positive or negative. Today's notes will focus on knowing when they are positive vs. negative based on the quadrant and what it means for the three trig functions.

Recall from earlier that in a right triangle the sine and cosine ratios were defined as:

$$\sin\theta = \frac{\text{opposite side length}}{\text{hypotenuse length}}$$
$$\cos\theta = \frac{\text{adjacent side length}}{\text{hypotenuse length}}$$
$$\tan\theta = \frac{\text{opposite side length}}{\text{adjacent side length}}$$

Example 10 Consider the **unit circle** shown below with an angle, θ, drawn in standard position.

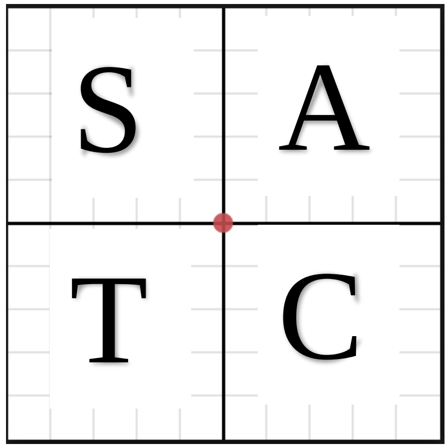


The Definition of the Sine, Cosine, and Tangent Functions

For an angle in standard position, whose terminal ray passes through the point (x, y) on the unit circle:

$$\sin(\theta) = the\ y-coordinate, \cos(\theta) = the\ x-coordinate, and\ \tan(\theta) = \frac{y-coordinate(\sin\theta)}{x-coordinate(\cos\theta)}$$

In addition, it is important to be able to determine the sign (positive or negative) of each of the three, basic Trigonometric functions for an angle whose terminal ray lies in each quadrant. Think of your x and y-coordinates in each quadrant…we use the following diagram to help us remember, and it's based on where the x and y-coordinates are positive or negative.



Reading counterclockwise from Quadrant 1 to Quadrant 4, we say: "All Students Take Calculus." In mathematical terms, the letters represent: A = All Functions, S = Sine, T = Tangent, C = Cosine. We refer to this as the **positivity chart**, since it tells us where functions are positive.

Quadrant 1, **A**ll the functions are positive since both x and y are positive in Quadrant 1.

Quadrant 2, the **S** represents the fact that Sin is positive here. This is because Sin is the y-coordinate, and in the 2nd quadrant, the x's are negative and the y's are positive.

Quadrant 3, the **T** represents the fact that Tangent is positive here. This is because in quadrant 3, both the x's and y's are negative, and $Tangent = \frac{Sin}{Cos}$, so a negative divided by a negative is a positive.

Quadrant 4, the **C** represents the fact that Cosine is positive here. This is because in quadrant 4, the x's are positive (and Cos is the x-coordinate) and y's are negative.

On its own, it may be confusing. Once we start putting it all together (by Weds.), it'll make way more sense.

So, this lesson's homework is to **memorize the Positivity Chart**, and then answer the questions below.

Practice:

1) Which quadrant would have Sin negative and Cos positive? 4

2) Where is Tan negative, but Sin positive? 2

3) In which quadrant is Cos < 0 and Tan > 0? 3

4) Name all quadrants where Tan is positive. 1 and 3

5) List any quadrant where Cos is negative. 2 and 3

Trig 2020 – Notes Day #5

Oftentimes we are asked in trigonometry for an "**exact value**." At these times, we are ignoring the calculator and looking for more specific values (often they are fractions). But where do these values come from? We get a little help from the special right triangles we used earlier.

Recall the two triangles: 30 – 60 – 90 and the 45 – 45 – 90. Both are right triangles, with specific values across from the angles. Now, how do these help us with our trigonometric functions? We'll create a table that utilizes these sides and their relationships.

	0	30	45	60	90
Sin	0	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	0
Tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined

A couple of things to note about the chart. Notice that Sin and Cos are opposite…Sin goes up, Cos goes down. They use the same values, but in reverse. And they are never larger than 1. The most important line is that of **Sin**…it drives the entire table! Know that one, and you're good! I say that because Cos is just Sin in reverse, and Tan is Sin/Cos…so the Tan values are derived from dividing the Sin value by the Cos value.

Notice that the Tan of 90 is Undefined. This is because dividing by zero leads to undefined values. If and when you get to the graphs of the Trig functions, you'll see how this undefined value is shown.

Side Note: I realize that $\sqrt{1} = 1$, but to make all the fractions in the table consistent, I added the radical sign. You won't see that in every rendition of this table, but I find it helps for memorizing purposes. (Hint, Hint!)

So, how do we use this along with everything else we learned? Let me give you some examples.

Example 11 What is the Sin of 135° ? If we entered this into a calculator, we would get a long, decimal answer. Our chart above, however, can lead us to an exact value answer, but we need to do some thinking.

First: Note that 135 is in the 2^{nd} quadrant, and therefore Sin is still positive (Positivity Chart from last lesson).

Second: We want to use a reference angle, so we subtract 180 – 135 = 45. So my reference angle is 45.

Third: Go to the exact value table above and find Sin 45.

Answer: $\frac{\sqrt{2}}{2}$

A lot goes into this seemingly simple question. Let's try another.

Example 12 Find the Cos of 210°.

So, 210 is in Quad 3, where Cos is negative. My reference angle is 30°. In my table, Cos of 30 = $\frac{\sqrt{3}}{2}$.

Therefore, Cos 210 = $-\frac{\sqrt{3}}{2}$. (The negative is there since we're in Quad 3.)

Homework #5: Try and solve the problems below using all that we've learned beforehand, and the exact value table.

1) Sin 225° = $-\frac{\sqrt{2}}{2}$

* Ref angle is 45

* Negative since it's in Quad 3

2) Tan 135° = -1

* Ref angle is 45

* Neg since it's in Quad 2

3) Cos 300° = $\frac{\sqrt{1}}{2} = \frac{1}{2}$

* Ref angle is 60

* Pos since it's in Quad 4

4) Sin 120° = $\frac{\sqrt{3}}{2}$

* Ref angle is 60

* Pos since it's in Quad 2

5) Cos -120° = $Cos\ 240 = -\frac{\sqrt{1}}{2}$

* Ref angle is 60

* Neg since it's Quad 3

6) Tan 315° = -1

* Ref angle is 45

* Neg since it's Quad 4

7) Sin 330° = $-\frac{\sqrt{1}}{2} = \frac{1}{2}$

* Ref angle is 30

* Neg since it's Quad 4

8) Cos 210° = $-\frac{\sqrt{3}}{2}$

* Ref angle is 30

* Neg since it's Quad 3

In general, you use the quadrant to decide whether it is positive or negative. The exact value comes from the table once you figure out the reference angle. Please realize that the exact value table is only good for a few angle values.

But what if we want to find an exact value for a different angle? What if we knew Sin or Cos but not both, could we find the other value…? The answer is yes, and the next page will begin to explain how we do this.

Trig 2020 – Notes Day #6

Not every angle we deal with falls on the exact value table, so we have other options to deal with these. Obviously the calculator will work, but it only returns decimal values. If we want an exact value (usually consisting of fractions and/or radicals), the calculator doesn't always help. One of our options is to use an old friend…since we are dealing with right triangles after all:

The Pythagorean Identity

Since each point on the unit circle must satisfy the equation $x^2 + y^2 = 1$,
we can now state what is known as the **Pythagorean Identity**.
For any angle, θ, $(cos\theta)^2 + (sin\theta)^2 = 1$. Shorthand Notation: $cos^2\theta + sin^2\theta = 1$

So how exactly would this work? Let me illustrate with the example below.

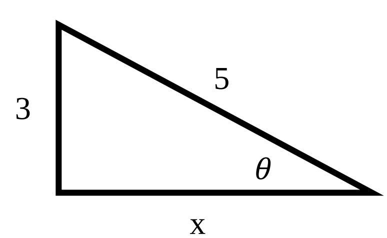
Example 13 An angle, θ, has a terminal ray that falls in the second quadrant on the unit circle. If it is known that $sin(\theta) = \frac{3}{5}$, determine the value of $cos(\theta)$.

Okay, since we know we're dealing with Quadrant 2, then Cos must return a negative value. But what is that value? We now have 2 options to find Cos, the Pythagorean Identity shown above, or make a triangle and use the Pythagorean theorem. Either way, Pythagoras wins!

Option 1: Pythagorean Identity

$cos^2\theta + sin^2\theta = 1$	Here's the general formula.
$cos^2\theta + (\frac{3}{5})^2 = 1$	Plug in the given value for Sin.
$cos^2\theta + \frac{9}{25} = 1$	Square top and bottom.
$cos^2\theta = \frac{16}{25}$	Start solving for Cos.
$cos\theta = -\frac{4}{5}$	Square root both sides. It's negative since we know it's in Quad 2.

Option 2: Make a triangle and label it with what we know. Since we were given Sin, and we know that $Sin = \frac{Opp}{Hyp}$, we can label two sides of the triangle. Using the Pythagorean formula we can find the missing side. We need the third side to find Cos, since it uses the A (adjacent side).



$3^2 + x^2 = 5^2$

$9 + x^2 = 25$

$x^2 = 16$, so $x = 4$ So, $Cos\,\theta = \frac{A}{H} = -\frac{4}{5}$

Both options work; you'll have to decide for yourself which one you prefer.

<u>Example 14</u> An angle, θ, has a terminal ray that falls in the *first quadrant* on the unit circle and $cos(\theta) = \frac{1}{3}$, determine the value of $sin(\theta)$ in **simplest radical form**.

Note the hint that our answer should have a radical with it? I'll use the Pythagorean Identity on this one to illustrate it one more time.

$cos^2\theta + sin^2\theta = 1$	State the Identity.
$(\frac{1}{3})^2 + sin^2\theta = 1$	Plug in what we know (Cos in this case).
$\frac{1}{9} + sin^2\theta = 1$	Square the Cos value.
$sin^2\theta = \frac{8}{9}$	Subtract from both sides to solve for $sin^2\theta$
$sin\theta = \frac{2\sqrt{2}}{3}$	Take the square root of both sides, simplify. It stays positive since we are in Quad 1.

<u>Example 15</u> If the terminal side of angle θ, in standard position **not on the unit circle**, passes through the point $(-5,12)$, what is the numerical value of $cos(\theta)$? *HINT: Since you are not in the unit circle the* $r \neq 1$. *Find the radius, which is the hypotenuse of the right triangle. Use Option 2 in this case.*

<u>Example 16</u> An angle, θ, has a terminal ray that falls in the third quadrant. If it is known that $\cos(\theta) = -.96$ determine the value of $\sin(\theta)$. **Show your work.** (I'd suggest Option 1 for this one)

Trig 2020 – Notes Day #7

Radian Angle Measurement

Just as distance can be measured in inches, feet, miles, centimeters, etcetera, rotations about a point can also be measured in different ways. A common unit of angle measurement that is an alternative to degrees is called the **radian**. One useful skill is the ability to convert between degrees and radians whenever it is convenient. Below is the method we use:

Degrees to Radians	**Radians to Degrees**
$\times \frac{\pi}{180}$	$\times \frac{180}{\pi}$

Example 17 Convert each of the following common angles in degrees into radians. Express your answers simplified in terms of π.

(a) $\theta = 90°$

$$90 \times \frac{\pi}{180} = \frac{90\pi}{180} = \frac{\pi}{2}$$

(b) $\theta = 120°$

$$120 \times \frac{\pi}{180} = \frac{120\pi}{180} = \frac{2\pi}{3}$$

(c) $\theta = 225°$

$$225 \times \frac{\pi}{180} = \frac{225\pi}{180} = \frac{5\pi}{4}$$

Notice that when the angle you are converting is less than 180 degrees, you are left with a proper fraction. Once you go over 180, you will get an improper fraction. To go the other way, simply replace the π with a 180 and then simplify.

Example 18 Convert each of the following common radian angles into degrees.

(a) $\theta = \frac{5\pi}{6}$

$$\frac{5\pi}{6} = \frac{5(180)}{6} = \frac{900}{6} = 150°$$

(b) $\theta = \frac{3\pi}{2}$

$$\frac{3\pi}{2} = \frac{3(180)}{2} = \frac{540}{2} = 270°$$

(c) $\theta = \frac{3\pi}{4}$

$$\frac{3\pi}{4} = \frac{3(180)}{4} = \frac{540}{4} = 135°$$

Again, notice the relationship between starting with a proper fraction in radians, and the resulting angle being less than 180 degrees.

Now, try the problems that follow on the next page:

Trig Assignment #7

A. Convert each of the following common degree angles to angles in radians. Express your answers in exact terms of pi.

(a) $30°$ (b) $45°$ (c) $60°$ (d) $180°$

(e) $300°$ (f) $135°$ (g) $270°$ (h) $330°$

B. Convert each of the following angles given in radians into an equivalent measure in degrees. Your answers will be integers.

(a) $\frac{2\pi}{3}$ (b) $-\frac{\pi}{2}$ (c) $\frac{11\pi}{4}$ (d) $-\frac{4\pi}{3}$